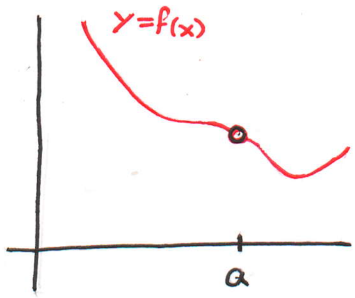
Lecture 4: Continuity

How can we mathematically define the idea that some curves can be drawn without lifting the pen from the paper and other curves cannot?

We will think of a function $f(x)$ as continuous if the graph $y=f(x)$ is one piece.

Ex. The following functions are discontinuous at $x=a$. What exactly goes wrong?

1)



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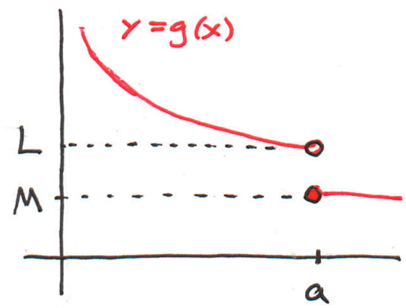

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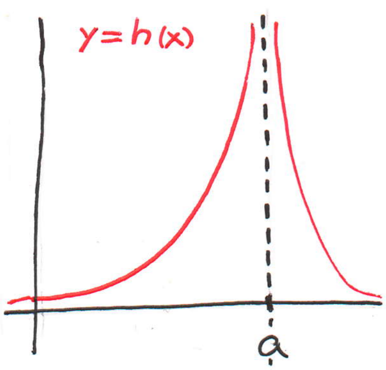

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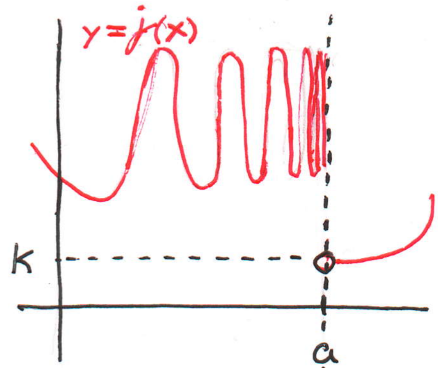


Solution:

1) Notice that $f(a)$ is undefined. However, $\lim_{x \to a} f(x)$ exists.

2) Here $g(a)$ is defined, but not $\lim_{x \to a} g(x)$:

$$\lim_{x \to a^-} g(x) = L \neq \lim_{x \to a^+} g(x) = g(a) = M.$$

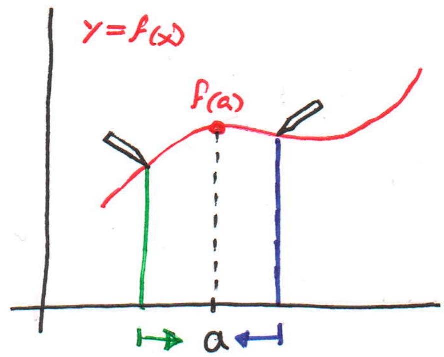
3) Neither $h(a)$, nor $\lim_{x \to a} h(x)$;

$$\lim_{x \to a} h(x) = +\infty.$$

4) Notice that $\lim_{x \to a^+} j(x) = k$, but $\lim_{x \to a^-} j(x) = \phi$

The left-hand limit diverges by oscilation.

For the curve $y = f(x)$ to be connected at $x = a$, the left part of the curve must merge with the right side of the curve.



Def: A function f is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

Def: A function f is continuous from the right at $x = a$ if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at $x = a$

$$\lim_{x \to a^-} f(x) = f(a)$$

Ex. Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$

(d) $f(x) = \lfloor x \rfloor$ greatest integer function.

Solution:

(a) $f(x) = \dfrac{x^2 - x - 2}{x-2}$ is a rational function

By the limit laws, $\lim_{x \to a} f(x) = f(a)$ unless $a = 2$.

Since $f(2) = \emptyset$, f is discontinuous at $a = 2$.

Notice, however, that $\lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2} =$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}} = \lim_{x \to 2} (x+1) = 3.$$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is defined everywhere

if $a \neq 0$ $\lim_{x \to a} f(x) = \lim_{x \to a} \dfrac{1}{x^2} = \dfrac{1}{a^2} = f(a)$

by the limit laws.

However $\lim_{x \to 0} \dfrac{1}{x^2} = \dfrac{1}{0^+} = +\infty$.

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

is clearly continuous at every $a \neq 2$.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x+1) = 3 \neq 1 = f(1)$$

Hence f is discontinuous at $a=2$.

(d) $f(x) = \lfloor x \rfloor$ works like this:

$f(3.1) = 3$ (round down), $f(5.9) = 5$,

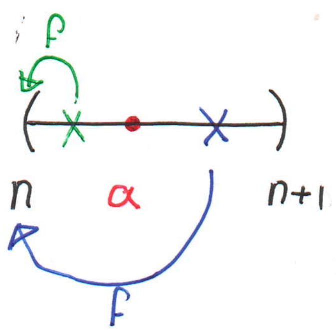
$f(-0.01) = -1$, $f(-5.9) = -6$.

Let $n < a < n+1$ be a number between two integers.

Then $\lim\limits_{x \to a^+} f(x) = f(a) = n$ and

$\lim\limits_{x \to a^-} f(x) = f(a) = n$

Thus f is continuous at $a \notin \mathbb{Z}$



If $a=n$, $\lim\limits_{x \to n^+} f(x) = n$ and $\lim\limits_{x \to n^-} f(x) = n-1$

Thus f is right-continuous at every a, but not left-continuous at every integer.

Ex. Show that the function $f(x) = 1 - \sqrt{1-x^2}$ is continuous on the interval $[-1,1]$

Solution:

If $-1<a<1$, $\lim\limits_{x\to a} \left(1-\sqrt{1-x^2}\right)$

$$= 1 - \lim_{x\to a} \sqrt{1-x^2} \quad \text{(by limit laws for } \pm\text{)}$$

$$= 1 - \sqrt{\lim_{x\to a} (1-x^2)} \quad \left(\lim \sqrt{\cdot} = \sqrt{\lim}\right)$$

$$= 1 - \sqrt{1-a^2} \quad \text{(by limit laws for } \times, \pm\text{)}$$

$$= f(a).$$

Similar calculations show that

$\lim\limits_{x\to -1^+} f(x) = 1 = f(-1)$ and $\lim\limits_{x\to 1^-} f(x) = f(1)$

Thm: If f and g are continuous at $x=a$ and c is a constant, then the following functions are also continuous at $x=a$.

1. $f+g$ 2. $f-g$ 3. cf

4. fg 5. $\frac{f}{g}$ if $g(a) \neq 0$

The proof is a simple application of the limit laws and the observations $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$.

<u>Remark</u>: Polynomials and Rational functions are continuous at every point where the function is defined.

If $p(x)$ is a polynomial

$\lim_{x \to a} p(x) = p(a)$ by limit laws for addition and multiplication.

If $R(x)$ is a rational map $R(x) = \frac{p(x)}{q(x)}$

$\lim_{x \to a} R(x) = \frac{p(a)}{q(a)}$ by limit laws for addition, multiplication, and division. (If $q(a) \neq 0$).

<u>Remark</u>: Continuity at $x = a$ amounts to the algebraic assersion that $\lim_{x \to a} f(x)$ gives the same result as simply plugging a into f.

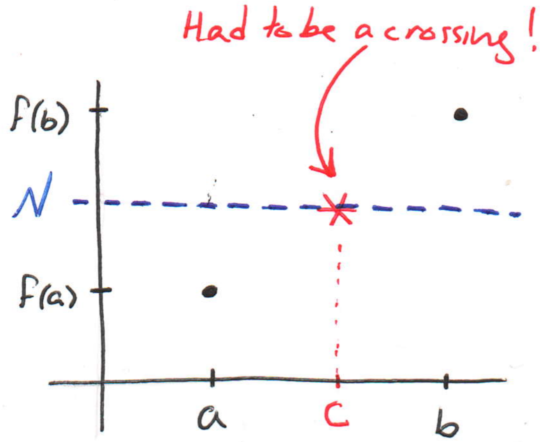
"Just plug in"!

The following theorem might appear to be stating the obvious, but its statement is intimately connected with the structure of the space $\mathbb{R}$.

Thm: (Intermediate Value Theorem)

Suppose f is continuous on the closed interval $[a,b]$ and N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$

Then there exists a number c in (a,b) such that $f(c) = N$.



Think of $y = f(x)$ as an invisible connected string in the plane. We only know the points $(a, f(a))$ and $(b, f(b))$. It appears intuitively obvious that at least one point of the string is in contact with the line $y = N$.

Ex. Show that the equation

$$4x^3-6x^2+3x-2=0.$$

has a root between 1 and 2.

Solution:

Let $p(x)=4x^3-6x^2+3x-2$. Then $p(x)$ is continuous but we don't see the graph of $p(x)$ (it is an invisible string to us).

However, $p(1) = -1 < 0$

$p(2) = 12 > 0$

Thus part of the curve is below the x-axis and another part is above the x-axis. By the intermediate value theorem, there must be at least one $1<c<2$ such that $p(c)=0$.

Ex. Show that the equation

$$\sqrt[3]{x} = 1-x$$

has a root (solution).

Solution:

It is often convenient to compare things to 0:

$$\sqrt[3]{x} = 1-x \quad \text{if and only if}$$

$$\sqrt[3]{x} + x - 1 = 0$$

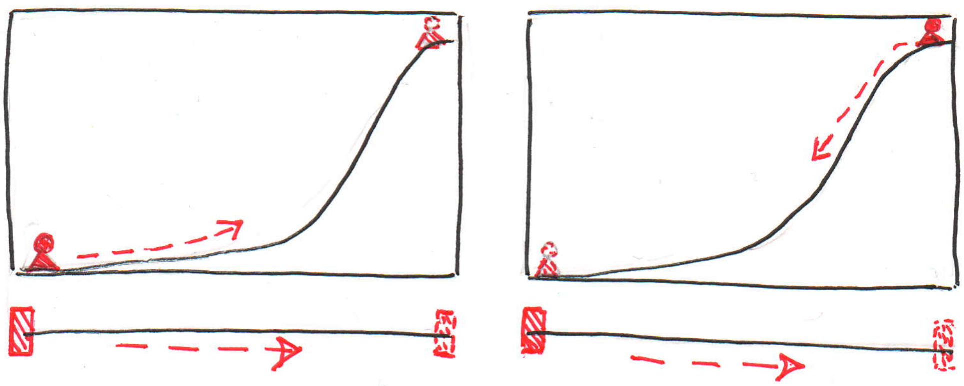
Let $f(x) = \sqrt[3]{x} + x - 1$. Clearly $f(x)$ is continuous.

Now $f(0) = \sqrt[3]{0} + 0 - 1 = -1 < 0$

$f(1) = \sqrt[3]{1} + 1 - 1 = 1 > 0$

Thus by the Intermediate Value Theorem, there is at least one value $0 < c < 1$ such that $f(c) = 0$.

Ex. A Tibetan monk leaves the monastery at 7:00 AM and takes his usual path to the top of the mountain, arriving at 7:00 PM. The following morning, he starts at 7:00 AM at the top and takes the same path back, arriving at the monastery at 7:00 PM. Show that there is a point on the path that the monk will cross exactly at the same time of day on both days.



Day 1 Day 2

Solution:

Can you demonstrate this without any fancy proofs?
How do you make the conclusion seem intuitive?

Mathematically, we can argue as follows:

Let $f(t)$ be the position of the monk on day 1 and $g(t)$ be the position of the monk on day 2.

Normalize time to be $0 \equiv$ 7:00 AM $1 \equiv$ 7:00 PM.

Normalize height to be $0 \equiv$ base of the mountain
$1 \equiv$ top of the mountain.

We wish to display that a solution to the equation

$$f(t) = g(t)$$

Must exist.

This equation is identical to

$$f(t) - g(t) = 0$$

Define $k(t) = f(t) - g(t)$.

Since the monk was moving without teleporting on day 1 and day 2, both $f(t)$ and $g(t)$ are continuous functions. Hence $k(t)$ is also continuous.

Notice that

$$k(0) = f(0) - g(0) = 0 - 1 = -1 < 0$$

$$k(1) = f(1) - g(1) = 1 - 0 = 1 > 0$$

Hence by the mean value theorem, there must be at least one moment of time $0 < t_0 < 1$ such that $k(t_0) = 0$.

This is the time when on both days, the monk stands on the same place on the mountain.